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FORCE FIELDS IN WHICH
CENTERS OF GRAVITY CAN BE DEFINED

by

Roy F. Reeves

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

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1951

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I. INTRODUCTION

A. Notation

In this thesis Euclidean spaces of one, two, and three dimensions are considered. These are designated by E_1 , E_2 , and E_3 respectively.

There is introduced in E_3 an orthogonal set of unit vectors \underline{U}_1 , \underline{U}_2 , and \underline{U}_3 which is to serve as a basis. The location of a point is usually specified by means of the position vector which is designated by \underline{R} . As there is little occasion to change the basis, the triple (V_1, V_2, V_3) is often written in place of the notation

$$\underline{V} = \underline{U}_1 V_1 + \underline{U}_2 V_2 + \underline{U}_3 V_3$$

in designating vectors. Thus, a point \underline{R}_1 is written as (X_1, Y_1, Z_1) whenever it is convenient to do so.

B. Definition of Mass and Mass Distributions in E_3

A point-mass, located at the point \underline{R}_1 , is defined as a real, positive number $M(\underline{R}_1)$. And a point-mass distribution is defined as a real, non-negative, single valued function $M(\underline{R})$ which is zero at all but a denumerable set of isolated points. Furthermore, the sum (finite or infinite) of the functional values of $M(\underline{R})$ in a bounded region is assumed

to be finite. A point-mass is often referred to as a particle, or mass-particle.

A piece-wise continuous distribution of mass is defined in terms of a non-negative, bounded, real, single valued density function, $D(\underline{R})$, which satisfies the following continuity condition: any bounded region can be divided into a finite number of sub-regions inside each of which $D(\underline{R})$ is continuous.

Let V be a bounded region containing a point-mass distribution with particles at the points \underline{R}_i , $i = 1, 2, 3, \dots$, along with a piece-wise continuous mass distribution with density function $D(\underline{R})$. The total mass in the region V is then defined to be M , where

$$M = \sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) dv.$$

C. Definition of a Force Field

A force field is defined to be a vector field, $\underline{F}(\underline{R})$, such that the force acting on a point-mass $M(\underline{R}_1)$ is given by

$$M(\underline{R}_1) \underline{F}(\underline{R}_1),$$

and such that the force acting on a mass that is piece-wise continuously distributed throughout a bounded region V is given by

$$\int_V D(\underline{R}) \underline{F}(\underline{R}) dv.$$

II. FIELDS OF FORCE ADMITTING OF A CENTER OF GRAVITY AS DEFINED BY ACZÉL AND FENYÖ¹

A. Center of Gravity as Defined by Aczél and Fenyő

Messrs. Aczél and Fenyő describe a force field as one which admits of a center of gravity if, corresponding to any system of point-masses placed in the field, a point called the center of gravity exists which satisfies the following conditions.

i) The forces acting on the system have as a resultant the force which would act on a particle placed at the center of gravity, if this particle had a mass equal to the total mass of the system.

ii) The mass system, if thought of as rigid, has no tendency to rotate about its center of gravity.

iii) The center of gravity of two point-masses lies on the straight line joining them.

In this paper the above definition is extended to require the existence of a point satisfying conditions i) and ii) for arbitrary mass distributions rather than for point-mass distributions only.

A mass system, which is contained entirely in a bounded region V , is now considered. If the mass system consists of particles $M(R_i)$, $i = 1, 2, 3, \dots$, and of a piece-wise continuous mass distribution with

¹Aczél, John and Fenyő, Stephen. On fields of force in which centers of gravity can be defined. *Hungarica Acta Mathematica*. 1,3: 53-60. 1948. (Original article not available, reprint used.)

density function $D(\underline{R})$ then conditions i), and ii) require the two equations:

$$2.1 \quad \left(\sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) dv \right) \underline{F}(\underline{R}_c) = \sum_{i=1} M(\underline{R}_i) \underline{F}(\underline{R}_i) + \int_V D(\underline{R}) \underline{F}(\underline{R}) dv,$$

and

$$2.2 \quad \left(\sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) dv \right) \underline{F}(\underline{R}_c) \times \underline{R}_c \\ = \sum_{i=1} M(\underline{R}_i) \underline{F}(\underline{R}_i) \times \underline{R}_i + \int_V D(\underline{R}) \underline{F}(\underline{R}) \times \underline{R} dv.$$

Here the symbol \times designates the vector or cross product, and \underline{R}_c designates the center of gravity.

The symbol $\underline{F}(\underline{R})$ is often written in an alternative form given by the equation

$$\underline{F}(\underline{R}) = (F_1(\underline{R}), F_2(\underline{R}), F_3(\underline{R})).$$

It is desired to find the most general function $\underline{F}(\underline{R})$ which satisfies the above two functional equations along with the following conditions. The first of these is that each of the functions $F_i(\underline{R})$, $i = 1, 2, 3$, is real, single valued, and continuous except possibly on a finite sum of point sets of the following types: 1) an isolated point, 2) a straight line, or 3) a plane. It is further assumed that, if \underline{R}_1 is a point at which $\underline{F}(\underline{R})$ is not continuous, then $\underline{F}(\underline{R})$ increases without bound as \underline{R} approaches \underline{R}_1 through points at which $\underline{F}(\underline{R})$ is continuous. If these conditions are met, no numerical value is assigned to \underline{F} at \underline{R}_1 , but \underline{F} is said to be infinite at \underline{R}_1 . The word "infinite" is used in this sense only.

Finally, it is assumed that each of the functions $F_i(\underline{U}_j X)$, $i, j = 1, 2, 3$, has a unique inverse, or is a constant. It is to be observed that, along any line, $\underline{F}(\underline{R})$ is either infinite at every point or is continuous with the exception of a finite number of isolated points where it is infinite.

The class of functions that is here considered is the same as that considered by Messrs. Aczél and Fenyő, though they did not state all of the above restrictions explicitly. The problem in E_3 is solved by a method similar to that used by Aczél and Fenyő to solve the problem in E_2 . Necessary restrictions on the force field will be found by considering mass-particles only, and then these conditions will be shown to be also sufficient for arbitrary mass distributions.

B. Line Distribution of Point-masses in E_3

1. Derivation of fundamental functional equations

It is supposed that $D(\underline{R}) \equiv 0$, and that the resulting point-mass system consists of n particles¹ located on the X -axis at the points $\underline{R}_i = \underline{U}_1 X_i$, $i = 1, 2, 3, \dots, n$. By condition iii), the center of gravity is a point on the X -axis, which is designated by $\underline{U}_1 X_c$. Equations 2.1 and 2.2 now reduce to

$$2.3 \quad \sum_{i=1}^n M(\underline{U}_1 X_i) \underline{F}(\underline{U}_1 X_i) = \sum_{i=1}^n M(\underline{U}_1 X_i) \underline{F}(\underline{U}_1 X_c),$$

and

¹For that part of the proof that establishes necessary conditions on \underline{F} , it is sufficient to let n be two.

$$\begin{aligned}
 2.4 \quad \sum_{i=1}^n M(\underline{U}_1 X_i) \underline{F}(\underline{U}_1 X_i) \times \underline{U}_1 X_c &= \sum_{i=1}^n M(\underline{U}_1 X_i) \underline{F}(\underline{U}_1 X_c) \times \underline{U}_1 X_c \\
 &= \sum_{i=1}^n M(\underline{U}_1 X_i) \underline{F}(\underline{U}_1 X_i) \times \underline{U}_1 X_i.
 \end{aligned}$$

If

$$\underline{F}(\underline{U}_1 X) = (F_1(\underline{U}_1 X), F_2(\underline{U}_1 X), F_3(\underline{U}_1 X)) = (G_1(X), G_2(X), G_3(X)),$$

then equations 2.3 and 2.4 become

$$2.3' \quad \sum_{i=1}^n M(\underline{U}_1 X_i) G_j(X_c) = \sum_{i=1}^n M(\underline{U}_1 X_i) G_j(X_i), \quad j = 1, 2, 3,$$

and

$$\begin{aligned}
 2.4' \quad \sum_{i=1}^n M(\underline{U}_1 X_i) G_j(X_i) X_c &= \sum_{i=1}^n M(\underline{U}_1 X_i) G_j(X_c) X_c \\
 &= \sum_{i=1}^n M(\underline{U}_1 X_i) G_j(X_i) X_i, \quad j = 2, 3.
 \end{aligned}$$

2. Two components of force assumed to be non-zero constants

It is supposed that $F_2(\underline{U}_1 X)$ and $F_3(\underline{U}_1 X)$ are constants, but are not both zero. In this event, equations 2.3' with $j = 2$, and 3 are obviously satisfied, and from either of the equations 2.4' it is clear that

$$2.5 \quad X_c = \frac{\sum_{i=1}^n M(\underline{U}_1 X_i) X_i}{\sum_{i=1}^n M(\underline{U}_1 X_i)}.$$

Substituting this expression into 2.3', with $j = 1$, gives

$$2.6 \quad G_1 \left(\frac{\sum_{i=1}^n M(U_1 X_i) X_i}{\sum_{i=1}^n M(U_1 X_i)} \right) = \frac{\sum_{i=1}^n M(U_1 X_i) G_1(X_i)}{\sum_{i=1}^n M(U_1 X_i)}.$$

Replacing $M(U_1 X_j) / \sum_{i=1}^n M(U_1 X_i)$ by M_j , $j = 1, 2, \dots, n$, in equation 2.6

shows that

$$2.7 \quad G_1 \left(\sum_{i=1}^n M_i X_i \right) = \sum_{i=1}^n M_i G_1(X_i),$$

where of course $\sum_{i=1}^n M_i = 1$. Putting $X_i = 0$ for $i \neq k$, results in the

n equations

$$2.8 \quad G_1(M_k X_k) = M_k G_1(X_k) + \sum_{i=1}^n ' M_i G_1(0), \quad k = 1, 2, \dots, n,$$

where the symbol ' indicates that the term of the summation that would involve M_k is left out. Substituting from 2.8 into 2.7 gives

$$2.9 \quad G_1 \left(\sum_{i=1}^n M_i X_i \right) = \sum_{k=1}^n G_1(M_k X_k) - \sum_{k=1}^n \sum_{i=1}^n ' M_i G_1(0).$$

If one subtracts

$$G_1(0) = \sum_{j=1}^n M_j G_1(0)$$

from both sides of 2.9, this equation becomes

$$2.10 \quad G_1\left(\sum_{i=1}^n M_i X_i\right) - G_1(0) = \sum_{k=1}^n (G_1(M_k X_k) - G_1(0)).$$

If

$$2.11 \quad M_k X_k = Z_k,$$

and if

$$2.13 \quad G_1(X) - G_1(0) = S_1(X),$$

equation 2.10 takes on the form

$$2.13 \quad S_1\left(\sum_{i=1}^n Z_i\right) = \sum_{i=1}^n S_1(Z_i).$$

By setting $Z_k = 0$, $K = 1, 2, \dots, n$, one sees that $S_1(0) = 0$, and one has only to set $Z_k = 0$, $K = 3, 4, \dots, n$, to see that

$$2.14 \quad S_1(Z_1 + Z_2) = S_1(Z_1) + S_1(Z_2).$$

Hence $S_1(X)$ must satisfy Cauchy's functional equation, and thus if $S_1(X)$ is continuous on some interval it is given by the equation

$$2.15 \quad S_1(X) = AX,$$

where A is an arbitrary constant.¹ By equation 2.12 it is seen that

$$2.16 \quad G_1(X) = AX + G_1(0) = AX + B,$$

or

$$2.17 \quad F_1(U_1X) = AX + B.$$

Thus, under the assumption that $F_2(U_1X)$ and $F_3(U_1X)$ are constants, not both zero, equations 2.3 and 2.4 can have a solution, which is continuous in some interval, only if it is of the form

$$2.18 \quad F(U_1X) = (AX + B, C, D),$$

where A, B, C, and D are constants. It is now shown by direct substitution that equation 2.18 actually satisfies equations 2.3 and 2.4.

Equation 2.3 requires

$$2.19 \quad \sum_{i=1}^n M(U_1X_i)(AX_i + B, C, D) \equiv \sum_{i=1}^n M(U_1X_i)(AX_c + B, C, D),$$

which reduces to

$$\sum_{i=1}^n M(U_1X_i)X_i \equiv \sum_{i=1}^n M(U_1X_i)X_c.$$

¹A proof of this well known result can be found in a book by Emile Picard. *Lecons sur quelques equations fonctionnelles*. Paris, Gauthier - Villars. 1928. p. 3.

This requires that

$$X_c = \frac{\sum_{i=1}^n M(\underline{U}_1 X_i) X_i}{\sum_{i=1}^n M(\underline{U}_1 X_i)},$$

which agrees with equation 2.5, the defining equation for X_c . Now equation 2.4 requires

$$\begin{aligned} 2.20 \quad \sum_{i=1}^n M(\underline{U}_1 X_i) (AX_i + B, C, D) \times (X_c, 0, 0) \\ \equiv \sum_{i=1}^n M(\underline{U}_1 X_i) (AX_i + B, C, D) \times (X_i, 0, 0), \end{aligned}$$

which becomes

$$X_c = \frac{\sum_{i=1}^n M(\underline{U}_1 X_i) X_i}{\sum_{i=1}^n M(\underline{U}_1 X_i)}.$$

It has been shown under the assumption that $F_2(\underline{U}_1 X)$ and $F_3(\underline{U}_1 X)$ are constants, not both zero, that

$$\underline{F}(\underline{U}_1 X) = (AX + B, C, D) \quad P_1, C_1.$$

is the only function satisfying the stated continuity conditions and satisfying equations 2.3 and 2.4 simultaneously. Furthermore, in this special case the center of gravity is given by

$$\underline{R}_c = \left(\frac{\sum_{i=1}^n M(\underline{U}_1 X_i) X_i}{\sum_{i=1}^n M(\underline{U}_1 X_i)}, 0, 0 \right).$$

If $A = 0$, all the vectors $\underline{F}(\underline{U}_1 X)$ are parallel. This case is designated by P_1 . However, if $A \neq 0$, the lines through $(X, 0, 0)$ with the same direction as $\underline{F}(\underline{U}_1 X)$ all pass through the point $(-\frac{B}{A}, -\frac{C}{A}, -\frac{D}{A})$. If such a point exists in a force field, then a force field is said to possess a source or center at this point, and the field is called a central field. The magnitude of \underline{F} at $(X, 0, 0)$ is proportional to the distance from $(X, 0, 0)$ to the source or center. This case is designated by C_1 .

3. Two components of force assumed to be identically zero

It is now assumed that $F_2(\underline{U}_1 X) \equiv F_3(\underline{U}_1 X) \equiv 0$. Under these conditions, equations 2.4' are obviously satisfied with no further restriction on $\underline{F}(\underline{U}_1 X)$. However, equations 2.3' require that

$$X_c = G_1^* \left(\frac{\sum_{i=1}^n M(\underline{U}_1 X_i) G_1(X_i)}{\sum_{i=1}^n M(\underline{U}_1 X_i)} \right),$$

where G_1^* is the inverse of G_1 . Quite clearly, if $G_1(X)$ is continuous for all X , then X_c is uniquely defined if and only if $G_1(X)$ is strictly monotone. This case is designated by L_1 . However, X_c is also uniquely defined if $G_1(X)$ is infinite at the one point X_i , provided $G_1(X)$ is strictly monotone

in each of the two intervals $-\infty < X < X_1$, and $X_1 < X < \infty$, and provided in addition that $G_1(X_1) \neq G_1(X_2)$ whenever $X_1 \neq X_2$. There is danger that the argument of G_1^* in the above equation will lie outside of the range of $G_1(X)$. This difficulty will be dealt with later. This case is designated by L_2 .

4. None of the components of the force are assumed to be constant

If $F_2(U_1 X)$ and $F_3(U_1 X)$ are not both identically constant, equations 2.3' give¹

$$2.21 \quad G_i^* \left(\frac{\sum_{k=1}^n M(U_1 X_k) G_1(X_k)}{\sum_{k=1}^n M(U_1 X_k)} \right) = G_j^* \left(\frac{\sum_{k=1}^n M(U_1 X_k) G_j(X_k)}{\sum_{k=1}^n M(U_1 X_k)} \right), \quad j \neq i.$$

If

$$2.22 \quad R_{ij}(X) = G_i(G_j^*(X)), \quad i \neq j,$$

then equations 2.21 can be written in the form

$$2.23 \quad \frac{\sum_{k=1}^n M(U_1 X_k) G_1(X_k)}{\sum_{k=1}^n M(U_1 X_k)} = R_{ij} \left(\frac{\sum_{k=1}^n M(U_1 X_k) G_j(X_k)}{\sum_{k=1}^n M(U_1 X_k)} \right), \quad i \neq j.$$

¹Similar functional equations were treated by John Aczél. On mean values. Bull. Am. Math. Soc. 54:392-410. 1948.

If one replaces $G_j(X_k)$ by $Z_{k,j}$, and X_k by $G_j^*(Z_{k,j})$, and then uses equation

2.22, this last equation becomes

$$2.24 \quad \frac{\sum_{k=1}^n M(\underline{U}_1 X_k) R_{1j}(Z_{k,j})}{\sum_{k=1}^n M(\underline{U}_1 X_k)} = R_{1j} \left(\frac{\sum_{k=1}^n M(\underline{U}_1 X_k) Z_{k,j}}{\sum_{k=1}^n M(\underline{U}_1 X_k)} \right), \quad i \neq j,$$

If one sets

$$\frac{M(\underline{U}_1 X_i)}{\sum_{k=1}^n M(\underline{U}_1 X_k)} = M_i, \quad i = 1, 2, \dots, n,$$

equation 2.24 can be rewritten in the form

$$2.24' \quad \sum_{k=1}^n M_k R_{1j}(Z_{k,j}) = R_{1j} \left(\sum_{k=1}^n M_k Z_{k,j} \right), \quad i \neq j.$$

These equations are of the same type as 2.7. Thus,

$$2.25 \quad R_{1j}(Z_{k,j}) = A_{1j} Z_{k,j} + B_{1j}, \quad i \neq j.$$

But, if the definitions of R_{1j} and $Z_{k,j}$ are applied, then equations 2.25 become

$$G_i(G_j^*(G_j(X))) = A_{1j} G_j(X) + B_{1j}, \quad i \neq j,$$

or

$$2.26 \quad G_i(X) = A_{ij}G_j(X) + B_{ij}, \quad i \neq j,$$

where A_{ij} , and B_{ij} are constants.

Next substituting from equations 2.4' into 2.3' gives

$$2.27 \quad G_i \left(\frac{\sum_{k=1}^n M(\underline{U}_1 X_k) G_i(X_k) X_k}{\sum_{k=1}^n M(\underline{U}_1 X_k) G_i(X_k)} \right) = \frac{\sum_{k=1}^n M(\underline{U}_1 X_k) G_i(X_k)}{\sum_{k=1}^n M(\underline{U}_1 X_k)}, \quad i = 2, 3.$$

or

$$2.28 \quad \frac{\sum_{k=1}^n M(\underline{U}_1 X_k) G_i(X_k) X_k}{\sum_{k=1}^n M(\underline{U}_1 X_k) G_i(X_k)} = G_i^* \left(\frac{\sum_{k=1}^n M(\underline{U}_1 X_k) G_i(X_k)}{\sum_{k=1}^n M(\underline{U}_1 X_k)} \right) \quad i = 2, 3.$$

After writing $G_i(X_k) = Z_{k,i}$, $X_k = G_i^*(Z_{k,i})$ and multiplying the result by

$$\frac{\sum_{k=1}^n M(\underline{U}_1 X_k) Z_{k,i}}{\sum_{k=1}^n M(\underline{U}_1 X_k)},$$

one sees that

$$2.29 \quad \frac{\sum_{k=1}^n M(\underline{U}_1 X_k) Z_{k,i} G_i^*(Z_{k,i})}{\sum_{k=1}^n M(\underline{U}_1 X_k)} = \frac{\sum_{k=1}^n M(\underline{U}_1 X_k) Z_{k,i}}{\sum_{k=1}^n M(\underline{U}_1 X_k)} G_i^* \left(\frac{\sum_{k=1}^n M(\underline{U}_1 X_k) Z_{k,i}}{\sum_{k=1}^n M(\underline{U}_1 X_k)} \right).$$

Setting $Z G_i^*(Z) = H_i(Z)$, and $M(U_1 X_k) / \sum_{i=1}^n M(U_1 X_i) = M_k$ changes equations

2.29 to the form

$$2.30 \quad \sum_{k=1}^n M_k H_i(Z_{k,i}) = H_i\left(\sum_{k=1}^n M_k Z_{k,i}\right), \quad i = 2, 3.$$

These are also equations of the same form as 2.7 and thus they have the solutions

$$H_i(t) = A_i t + B_i, \quad i = 2, 3.$$

After using the definitions of H_i , one sees that

$$2.31 \quad G(X) = \frac{B_i}{X - A_i} = \frac{1}{C_i X + D_i}, \quad i = 2, 3.$$

For $i = 3$, and $j = 2$, equation 2.26 is

$$2.32 \quad G_3(X) = A_{32} G_2(X) + B_{32},$$

which when combined with 2.31 gives

$$2.33 \quad \frac{1}{C_3 X + D_3} = \frac{A_{32}}{C_2 X + D_2} + B_{32}.$$

Equation 2.26 implies that if one of the functions $G_2(X)$ or $G_3(X)$ is not identically constant, then the other is not constant. Thus neither C_2 nor C_3 is zero, and hence equation 2.33 implies

$$2.34 \quad B_{32} = 0.$$

Combining the results of equations 2.26, 2.31, and 2.34 with the definition of G_i gives

$$2.35 \quad F_1(U_1X) = \frac{A_{12}}{C_2X + D_2} + B_{12} = \frac{S + TX}{C_2X + D_2},$$

$$2.36 \quad F_2(U_1X) = \frac{1}{C_2X + D_2},$$

and

$$2.37 \quad F_3(U_1X) = \frac{E}{C_2X + D_2}.$$

It remains to be shown that the function

$$F(U_1X) = \left(\frac{S + TX}{C_2X + D_2}, \frac{1}{C_2X + D_2}, \frac{E}{C_2X + D_2} \right) \quad P_2, C_2.$$

actually satisfies equations 2.3 and 2.4. This is equivalent to showing that G_1 , G_2 , and G_3 , as defined in this case, satisfy equations 2.3', and

2.4'. First, G_1 is substituted into 2.3' with $i = 1$. This gives

$$\sum_{i=1}^n M(U_1X_i) \left(\frac{A_{12}}{C_2X_i + D_2} + B_{12} \right) = \sum_{i=1}^n M(U_1X_i) \left(\frac{A_{12}}{C_2X_i + D_2} + B_{12} \right).$$

This equation reduces to

$$\sum_{i=1}^n M(\underline{U}_1 X_i) \frac{A_{12}}{C_2 X_c + D_2} = \sum_{i=1}^n M(\underline{U}_1 X_i) \frac{A_{12}}{C_2 X_1 + D_2}.$$

Thus, if G_1 satisfies 2.3', so then do G_2 , and G_3 . This last equation is satisfied if and only if

$$2.38 \quad X_c = \frac{\sum_{k=1}^n M(\underline{U}_1 X_k) \prod_{i=1}^n (C_2 X_1 + D_2) - D_2 \sum_{k=1}^n M(\underline{U}_1 X_k) \prod_{i=1}^n (C_2 X_1 + D_2)'}{C_2 \sum_{k=1}^n M(\underline{U}_1 X_k) \prod_{i=1}^n (C_2 X_1 + D_2)'},$$

where the symbol ' indicates that the term of the product that would involve X_k is left out. Next, G_3 is substituted into 2.4' with $j = 3$. This gives

$$\sum_{i=1}^n M(\underline{U}_1 X_i) \frac{EX_c}{C_2 X_c + D_2} = \sum_{i=1}^n M(\underline{U}_1 X_i) \frac{EX_1}{C_2 X_1 + D_2}.$$

Clearly, if this equation is satisfied by the value of X_c defined above, then 2.4' is also satisfied by G_2 . This last equation can be written in a different form.

$$\sum_{i=1}^n M(\underline{U}_1 X_i) \left(\frac{E}{C_2} - \frac{\frac{D_2 E}{C_2}}{C_2 X_c + D_2} \right) = \sum_{i=1}^n M(\underline{U}_1 X_i) \left(\frac{E}{C_2} - \frac{\frac{D_2 E}{C_2}}{C_2 X_1 + D_2} \right).$$

This is equivalent to

$$\sum_{i=1}^n M(\underline{U}_1 X_i) \frac{1}{C_2 X_c + D_2} = \sum_{i=1}^n M(\underline{U}_1 X_i) \frac{1}{C_2 X_1 + D_2},$$

which, it has been shown, is satisfied for X_c as given by equation 2.38.

This completes the proof of the fact that, if $F_2(\underline{U}_1 X)$ and $F_3(\underline{U}_1 X)$ are not both identically constant, and if \underline{F} satisfies the required continuity conditions along the X -axis, then

$$2.39 \quad \underline{F}(\underline{U}_1 X) = \left(\frac{S + TX}{C_2 X + D_2}, \frac{1}{C_2 X + D_2}, \frac{E}{C_2 X + D_2} \right). \quad P_2, C_2.$$

If $T = 0$, then \underline{F} is a parallel field which is infinite at $X = -\frac{D_2}{C_2}$.

The magnitude of \underline{F} is given by

$$2.40 \quad \left| \underline{F}(\underline{U}_1 X) \right| = \frac{1}{C_2} \frac{\sqrt{E^2 + S^2 + 1}}{X + \frac{D_2}{C_2}}.$$

That is, the magnitude of the force at $(X, 0, 0)$ is inversely proportional to the distance from $(X, 0, 0)$ to any plane that contains the point

$(-\frac{D_2}{C_2}, 0, 0)$ but not the point $(X, 0, 0)$.

This case is designated by P_2 .

If $T \neq 0$, $\underline{F}(\underline{U}_1 X)$ has a source located at $(\frac{-S}{T}, \frac{-1}{T}, \frac{-E}{T})$, and the magnitude of $\underline{F}(\underline{U}_1 X)$ is given by

$$2.41 \quad \underline{F}(\underline{U}_1 X) = \frac{T}{C_2} \frac{\sqrt{\left(X + \frac{S}{T}\right)^2 + \frac{1}{T^2} + \left(\frac{E}{T}\right)^2}}{X + \frac{D_2}{C_2}}.$$

5. Summary of results obtained by assuming a line distribution of particles

It is important to realize that considering the case in which all mass-particles are along the X-axis has not restricted the results seriously, for any line in the field could have been taken as the X-axis. It was shown that if \underline{F} satisfies the stated continuity conditions, and if there exists a point satisfying conditions i), ii) and iii), then one of the three following conditions holds: \underline{F} is continuous everywhere, \underline{F} is discontinuous at one point only, where it is infinite, or it is infinite everywhere. In case $\underline{F}(\underline{R})$ is continuous at every point of a given line, it is of type C_1 , P_1 , or L_1 . Otherwise $\underline{F}(\underline{R})$ is of type C_2 , P_2 , or L_2 . The four cases P_1 , P_2 , C_1 , and C_2 are tabulated below, and are illustrated by Figures 1, 2, 3, and 4 respectively.

$$C_1: \quad \underline{F}(\underline{U}_1 X) = (AX + B, C, D).$$

The source or center for this case is at the point $\theta = -\left(\frac{B}{A}, \frac{C}{A}, \frac{D}{A}\right)$.

$$C_2: \quad \underline{F}(\underline{U}_1 X) = \left(\frac{S + TX}{C_2 X + D_2}, \frac{1}{C_2 X + D_2}, \frac{E}{C_2 X + D_2}\right).$$

The source or center in this case is at the point $\theta = -\left(\frac{S}{T}, \frac{1}{T}, \frac{E}{T}\right)$.

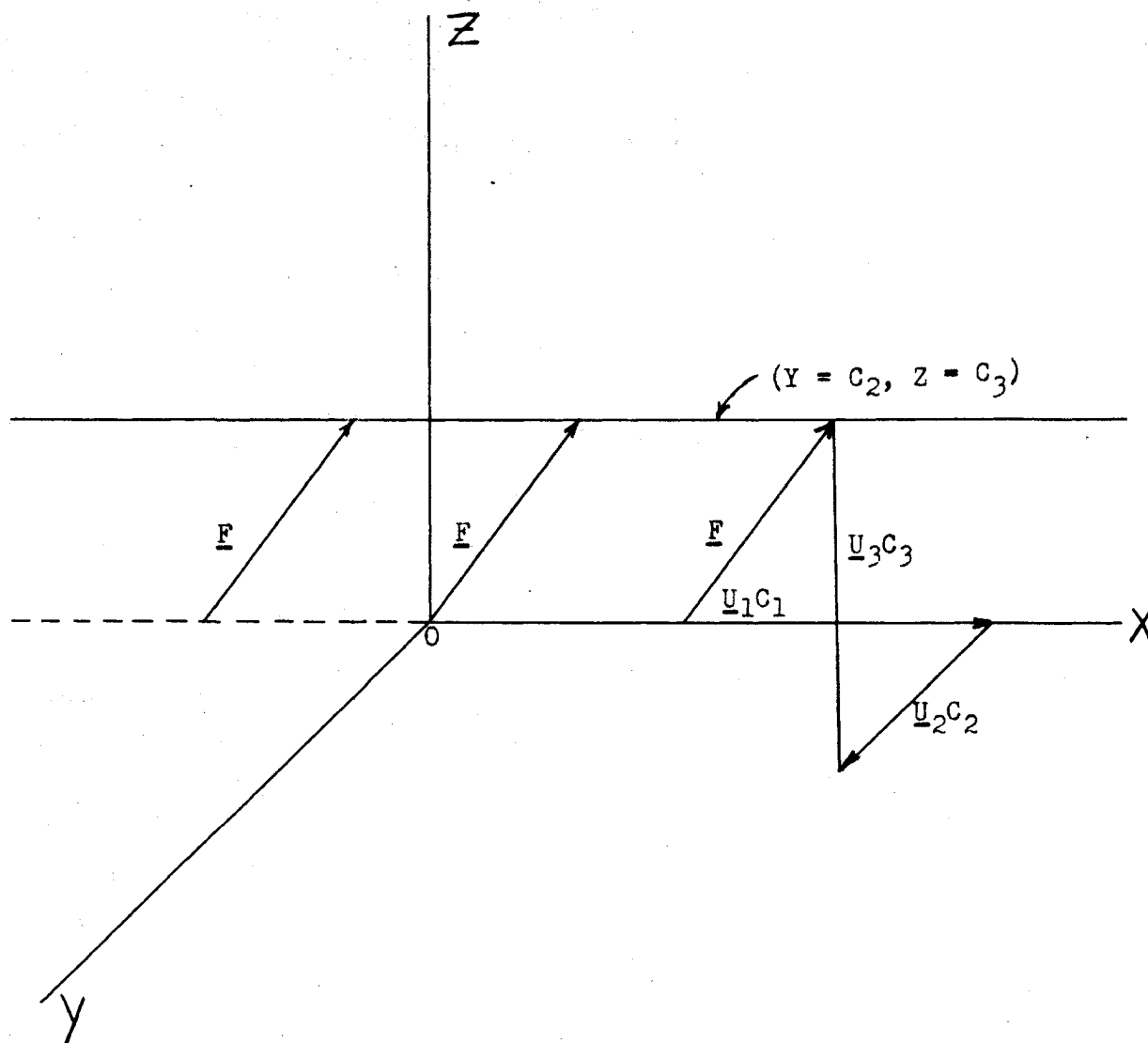


Figure 1.

Case P_1 : A Parallel Force Field

Values of the vector function that define the force field are shown for various values of X . The equation for \underline{F} in this case is

$$\underline{F}(\underline{U}_1 X) = (C_1, C_2, C_3).$$

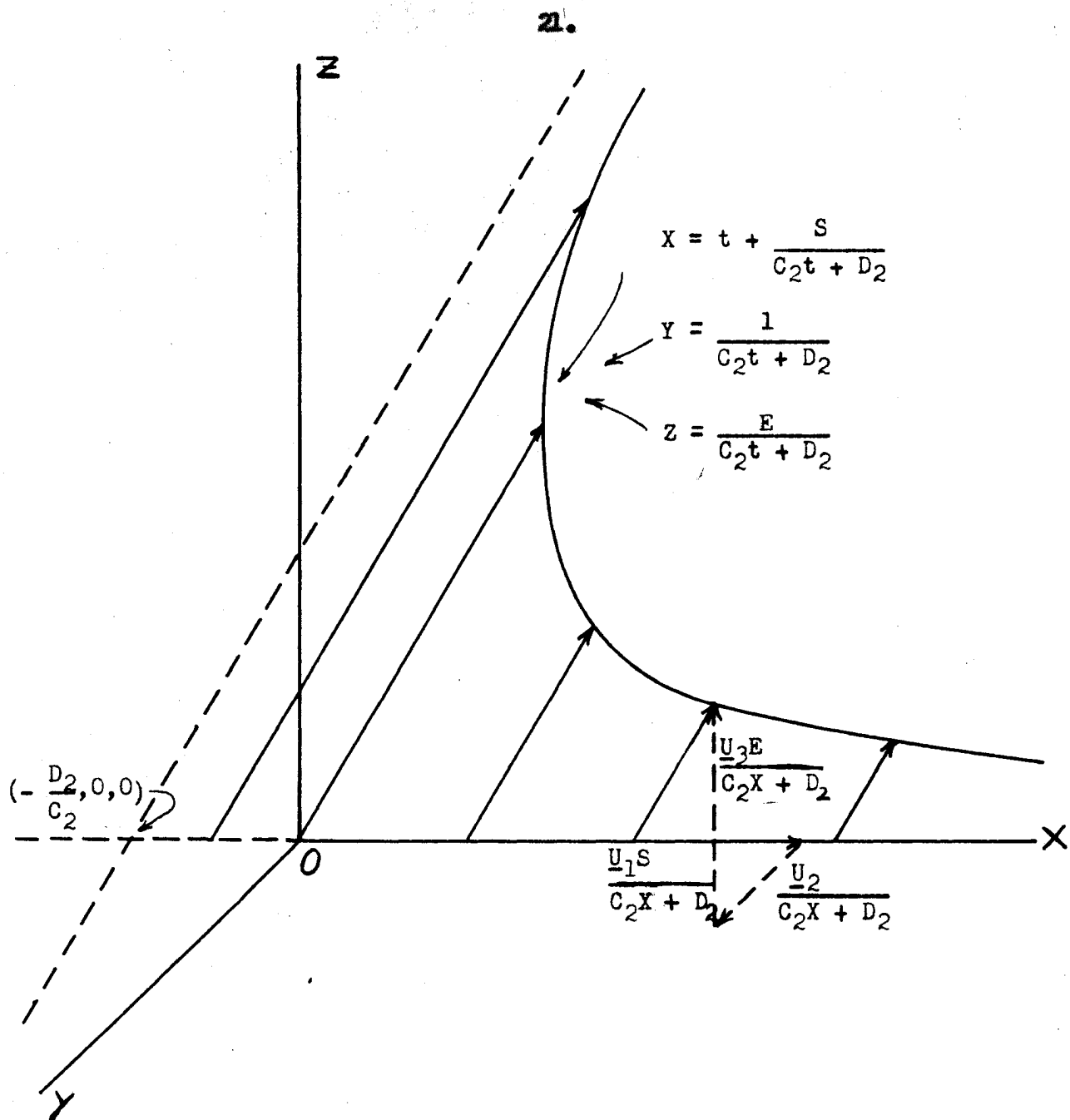


Figure 2.

Case P_2 : A Parallel Force Field

Values of the vector function that define the force field are shown for various values of X . The equation for \underline{F} in this case is

$$\underline{F}(\underline{U}_1 \underline{x}) = \frac{1}{C_2 X + D_2} (S, 1, E)$$

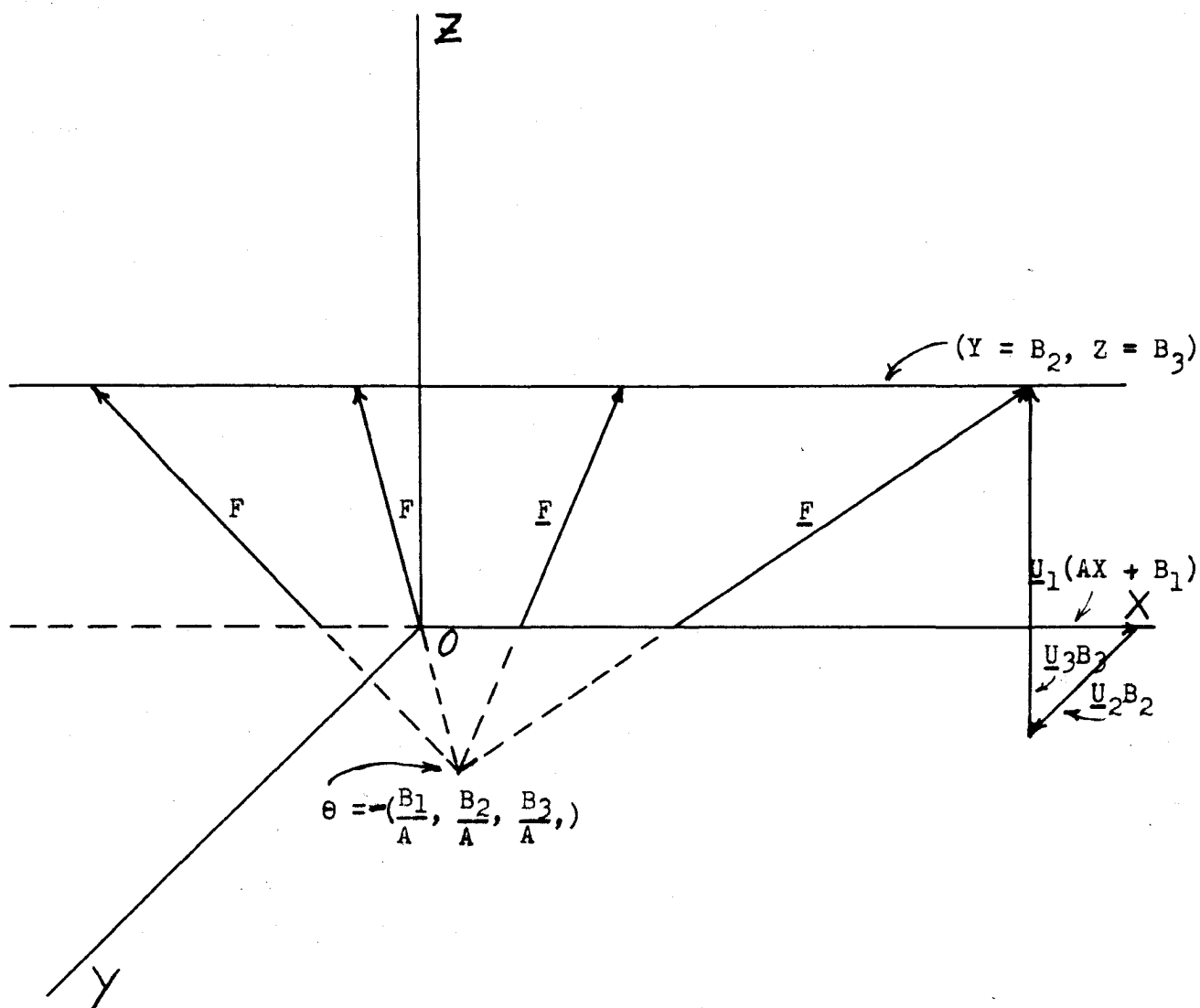


Figure 3

Case C_1 : A Central Force Field

Values of the vector function that define the force field are shown for various values of X . The equation for \underline{F} in this case is

$$\underline{F}(U_1X) = (AX + B_1, B_2, B_3)$$

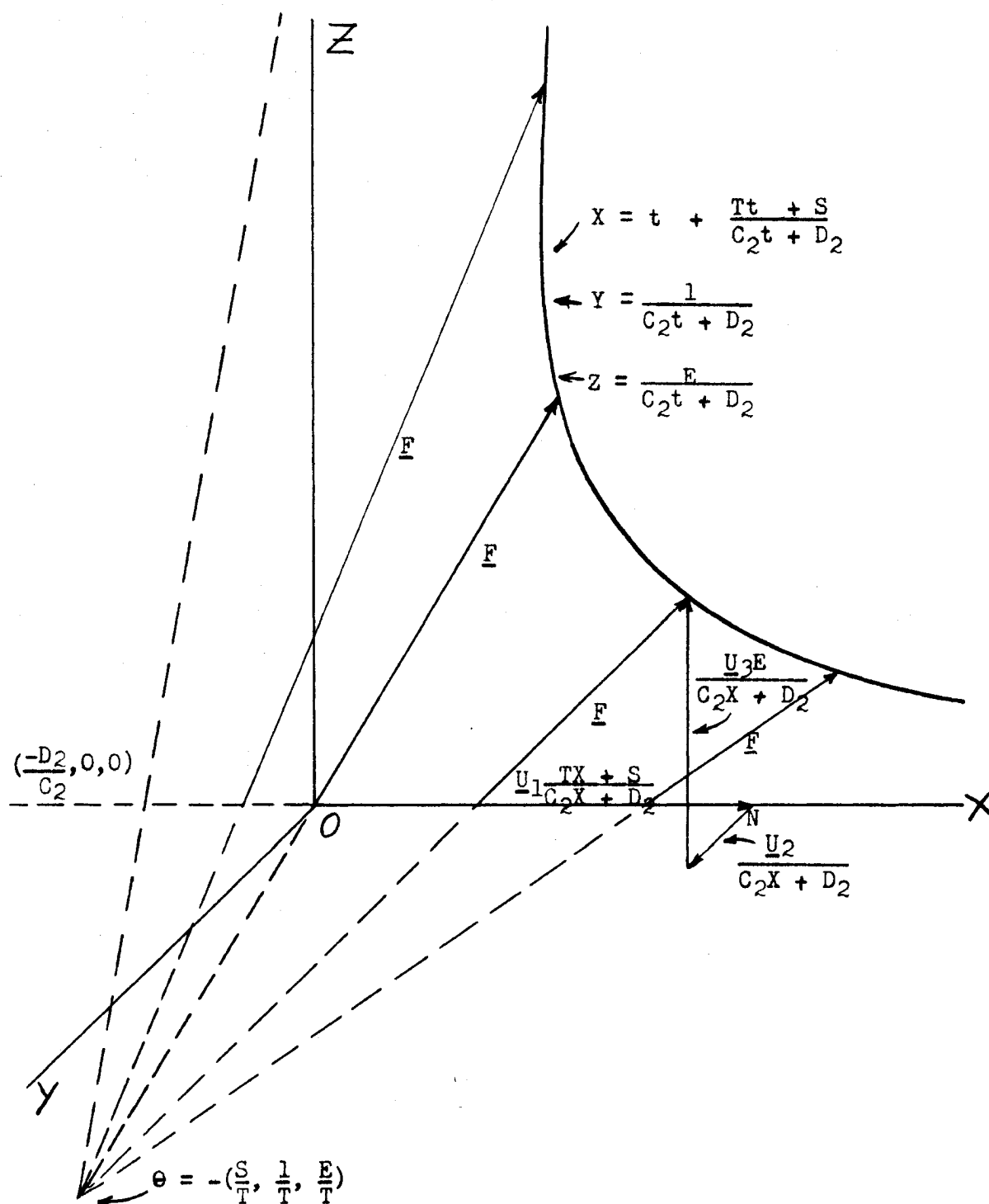


Figure 4

Case C_2 : A Central Force Field

Values of the vector function that define the force field are shown for various values of X . The equation for \underline{F} in this case is

$$\underline{F}(U_1 X) = \frac{1}{C_2X + D_2} (TX + S, 1, E).$$

$$F_1: \quad \underline{F}(\underline{U}_1 X) = (B, C, D).$$

$$P_2: \quad \underline{F}(\underline{U}_1 X) = \left(\frac{S}{C_2 X + D_2}, \frac{1}{C_2 X + D_2}, \frac{E}{C_2 X + D_2} \right).$$

C. Arbitrary Distribution of Point-masses in E_3

1. Classification of the force fields

It is clear from the above results, that if there exists a force field $\underline{F}(\underline{R})$ that satisfies the stated continuity conditions, and if there exists a point satisfying condition i), ii), and iii), then on any line, on which \underline{F} is not everywhere infinite, the forces are all parallel, or their directions point toward a common point called the source or center.

2. Force fields consisting of parallel forces only

First, the case is considered in which the forces are all parallel on every line on which \underline{F} is not infinite at each point. Thus, on such lines, the force is zero or infinite at not more than one point, for the force field on each must be of type P_1 , P_2 , L_1 , or L_2 . If two lines intersect at a point where the force is neither zero nor infinite, then the forces on these two lines are all parallel as required by the force at the common point of the two lines. Thus the force field consists only of parallel forces, since the direction of \underline{F} at points where \underline{F} is zero or

infinite is defined in such a way as to be consistent with this.

3. Force fields possessing a source or center

The case in which there exist lines on which the forces are finite but not all parallel is considered next. In Figure 5, e is such a line, and θ is the assumed center for the forces on e . On e , the force field \underline{F} must be of type C_1 , or C_2 , and can thus be infinite at not more than one point. Also, \underline{R}_1 , \underline{R}_2 , and \underline{R}_3 are three points on e at which \underline{F} is finite but different from zero, and \underline{R} is any point, different from θ , that is not on e itself. Further, it is assumed that the force at \underline{R} is finite but different from zero. Now e' is the line that passes through the points \underline{R} and \underline{R}_1 , e'' is the line that passes through the points \underline{R} and \underline{R}_2 , and e''' is the line that passes through the points \underline{R} and \underline{R}_3 . The forces at \underline{R}_1 , \underline{R}_2 , and \underline{R}_3 must have directions which point toward θ as indicated by the arrows at these points. It is easy to see that the forces can not be parallel on as many as two of the lines e' , e'' , and e''' , for their directions would not agree as they must at the point \underline{R} . First, it is supposed that the forces on e' are all parallel, but that the forces on e'' , and e''' possess centers. The direction of \underline{F} at \underline{R}_2 requires the center for e'' to lie along the line through θ and \underline{R}_2 ; the direction of \underline{F} at \underline{R}_3 indicates that the center for e''' must lie along the line through θ and \underline{R}_3 ; and the direction of \underline{F} at \underline{R} requires that the forces on these two lines have a common center. The common center, of course, is θ . Hence, it is clear that the forces along e' are not all parallel. This is a contradiction, and thus the forces

26.

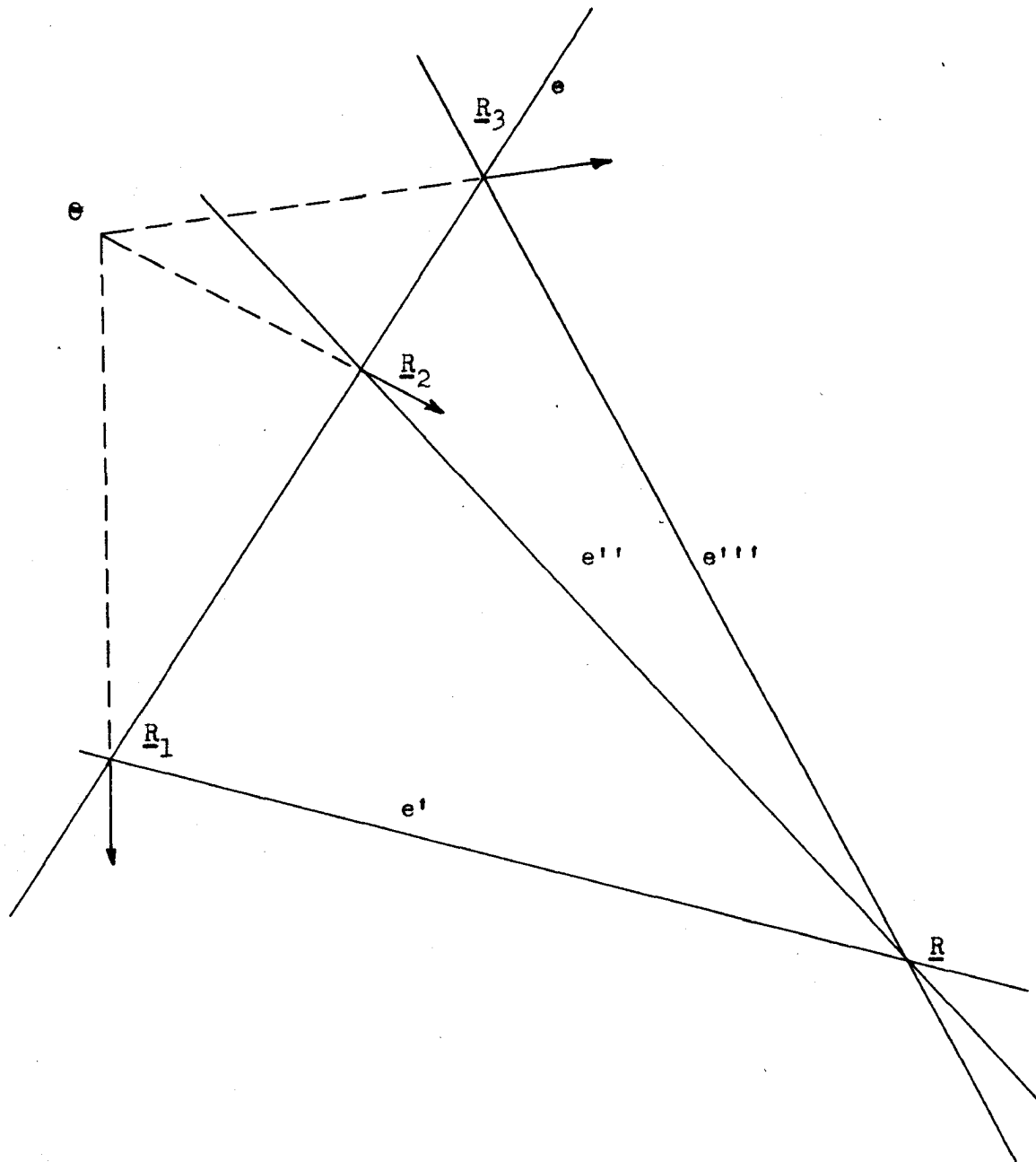


Figure 5.

This is a diagram illustrating the case in which there are lines along which the forces are not all parallel.

along e' do possess a center which, in fact, must be θ . Thus, the forces on each of the three lines possess a common center. Now \underline{R} is any point at which \underline{F} is finite but different from zero, so it has been proved that, if in E_3 there are lines along which the forces are not all parallel, then at each point \underline{R} in space the direction of \underline{F} is toward one fixed point, the source or center of the force field. Of course, the direction of \underline{F} is defined at a point where \underline{F} is zero or infinite in such a way as to be consistent with this statement.

4. Central force fields which are everywhere finite

It is next supposed that the field is central but that at no point in E_3 is the force infinite. Then, the force field is of type C_1 on every line that does not pass through the source θ . Along such lines the force is proportional to the distance from the center θ . The proportionality constants for two intersecting lines are the same, and thus there is just one constant for all points in E_3 . Hence, in a coordinate system with θ as the origin,

$$2.42 \quad \underline{F}(\underline{R}) = A\underline{R}, \quad \text{CI.}$$

where A is an arbitrary constant. This case is designated by CI.

5. Central force fields which become infinite at one or more points

Next it is assumed that the forces are central but that the field is infinite at one point \underline{R}_1 . Then there must be a second point \underline{R}_2 at which

$\underline{F}(\underline{R})$ is infinite, for otherwise the field would be finite on any line not passing through \underline{R}_1 . The field would then be of type CI at all points other than \underline{R}_1 . Such a case is contradictory, for the absolute value of \underline{F} must be arbitrarily large in every neighborhood of \underline{R}_1 . Hence, on the line through \underline{R}_1 and \underline{R}_2 , $\underline{F}(\underline{R})$ is infinite at every point. By an argument similar to that above it is clear that there must be a third point at which the force becomes infinite. Since $\underline{F}(\underline{R})$ is infinite at every point of every line that passes through \underline{R}_3 and also intersects the line through \underline{R}_1 and \underline{R}_2 , then \underline{F} is infinite at every point in the plane P defined by \underline{R}_1 , \underline{R}_2 , and \underline{R}_3 . However, it is to be noted that, if \underline{F} is assumed to be infinite at some point \underline{R}_4 that is not in P, then \underline{F} is infinite at every point of every line which passes through \underline{R}_4 but which is not parallel to P. Thus, under this assumption, \underline{F} is nowhere defined. This proves that \underline{F} is infinite only on the plane P. Let \underline{R} be any point on a line which passes through Θ but which is not parallel to P. Therefore, the force at \underline{R} is directly proportional to the distance from \underline{R} to Θ , and inversely proportional to the distance from \underline{R} to P. The proportionality constants for two intersecting lines are the same, and hence there is just one proportionality constant for the whole space. If the X-axis is taken as that line which is perpendicular to P and passes through Θ , and if the origin is taken in the plane P, then

$$2.43 \quad \underline{F}(\underline{R}) = B \left(\frac{X-D}{X}, \frac{Y}{X}, \frac{Z}{X} \right), \quad \text{CII.}$$

where D is the distance from O to P, and where B is an arbitrary constant. This case is designated by CII.

6. Parallel force fields which are everywhere finite

It is now assumed that $\underline{F}(\underline{R})$ is a parallel field, but that \underline{F} is not infinite at any point in E_3 . In this event, $\underline{F}(\underline{R})$ is of type P_1 on every line in E_3 . The force is the same at every point of any given line and hence is the same at every point of any two intersecting lines. Thus, it is constant throughout E_3 , that is

$$2.45 \quad \underline{F}(\underline{R}) = (C_1, C_2, C_3), \quad \text{PI.}$$

where C_1 , C_2 , and C_3 are arbitrary constants. This case is designated by PI.

7. Parallel fields which are infinite at one or more points

Lastly, the case is considered in which the forces are all parallel, but become infinite at one or more points. If there is just one point \underline{R}_1 at which $\underline{F}(\underline{R})$ is infinite, then on any line not passing through \underline{R}_1 , the field is of type P_1 . This is contradictory for it requires $\underline{F}(\underline{R})$ to be of type PI at all points other than \underline{R}_1 . Thus $\underline{F}(\underline{R})$ is infinite at a second point \underline{R}_2 , and in fact by a similar argument it is easily shown to be infinite at a third point \underline{R}_3 . As in case CII, it is easy to see that $\underline{F}(\underline{R})$ is infinite at every point in the plane P determined by the points \underline{R}_1 , \underline{R}_2 , and \underline{R}_3 , but is continuous elsewhere. On every line not parallel to P , $\underline{F}(\underline{R})$ is of type P_2 . This means that at any point on a given line the force is inversely proportional to the distance from the point to P . Necessarily, the proportionality constant is the same for all points in

E_3 as required by the fact that it is the same for all points of a pair of intersecting lines. Hence, if the X-axis is taken perpendicular to P and the origin is taken in P, then

$$2.46 \quad \underline{F}(\underline{R}) = \frac{1}{X} (D_1, D_2, D_3), \quad \text{PII.}$$

where D_1 , D_2 , and D_3 are arbitrary constants. This case is designated by PII.

8. Summary of results obtained by considering point-mass distribution only

At this stage, it has been shown that if a force field $\underline{F}(\underline{R})$ satisfies the stated continuity conditions, and if there exists a point satisfying conditions i), ii) and iii) for an arbitrary mass distribution, then \underline{F} must be one of the following types.

$$2.47 \quad \text{CI:} \quad \underline{F}(\underline{R}) = A\underline{R}.$$

$$2.48 \quad \text{CII:} \quad \underline{F}(\underline{R}) = B\left(\frac{X+D}{X}, \frac{Y}{X}, \frac{Z}{X}\right).$$

$$2.49 \quad \text{PI:} \quad \underline{F}(\underline{R}) = (C_1, C_2, C_3).$$

$$2.50 \quad \text{PII:} \quad \underline{F}(\underline{R}) = \frac{1}{X} (D_1, D_2, D_3).$$

It remains to be shown that these force fields actually satisfy the required conditions. It is, however, clear that they satisfy the required continuity conditions.

D. Arbitrary Mass Distributions in E_3

An arbitrary mass system, of total mass M , is now considered. This system is assumed to be contained entirely in a bounded region V , and to consist of particles of mass $M(\underline{R}_i)$, $i = 1, 2, \dots$, as well as of piece-wise continuously distributed mass of density function $D(\underline{R})$. It is now proved by direct substitution that $\underline{F}(\underline{R})$ may be of type CI, CII, PI, or PII.

First, $\underline{F}(\underline{R})$ is assumed to be of type CI, that is,

$$2.51 \quad \underline{F}(\underline{R}) = A\underline{R}.$$

This last expression is substituted into equation 2.1 to obtain

$$2.52 \quad \left(\sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) d\underline{v} \right) A\underline{R}_c = \sum_{i=1} M(\underline{R}_i) A\underline{R}_i + \int_V D(\underline{R}) A\underline{R} d\underline{v},$$

or

$$2.53 \quad M\underline{R}_c = \sum_{i=1} M(\underline{R}_i) \underline{R}_i + \int_V D(\underline{R}) \underline{R} d\underline{v}.$$

Thus equation 2.1 is satisfied if and only if

$$2.54 \quad \underline{R}_c = \frac{\sum_{i=1} M(\underline{R}_i) \underline{R}_i + \int_V D(\underline{R}) \underline{R} d\underline{v}}{M}.$$

Next, \underline{F} as given by 2.51, is substituted into 2.2. This gives

$$2.55 \quad M \underline{A} \underline{R}_c \times \underline{R}_c = \sum_{i=1} M(\underline{R}_i) \underline{A} \underline{R}_i \times \underline{R}_i + \int_V D(\underline{R}) \underline{A} \underline{R} \times \underline{R} d\underline{v}.$$

Both sides of this equation are identically zero. Thus it is seen that a force field of type CI does satisfy the stated conditions.

Secondly, it is supposed that $\underline{F}(\underline{R})$ is of type CII, that is, that

$$2.56 \quad \underline{F}(\underline{R}) = B \left(\frac{X-D}{X}, \frac{Y}{X}, \frac{Z}{X} \right).$$

By substituting this last expression into equation 2.1 it is seen that

$$2.57 \quad MB \frac{X_c - D}{X_c} = \sum_{i=1} M(\underline{R}_i) B \frac{X_i - D}{X_i} + \int_V D(\underline{R}) B \frac{X-D}{X} d\underline{v},$$

$$2.58 \quad MB \frac{Y_c}{X_c} = \sum_{i=1} M(\underline{R}_i) B \frac{Y_i}{X_i} + \int_V D(\underline{R}) B \frac{Y}{X} d\underline{v},$$

and

$$2.59 \quad MB \frac{Z_c}{X_c} = \sum_{i=1} M(\underline{R}_i) B \frac{Z_i}{X_i} + \int_V D(\underline{R}) B \frac{Z}{X} d\underline{v}.$$

These equations are satisfied if

$$2.60 \quad X_c = \frac{M}{\sum_{i=1} \frac{M(\underline{R}_i)}{X_i} + \int_V \frac{D(\underline{R})}{X} d\underline{v}},$$

$$2.61 \quad Y_c = \frac{\sum_{i=1}^n \frac{M(R_i)}{X_i} \frac{Y_i}{X_i} + \int_V \frac{D(R)}{X} \frac{Y}{X} dv}{\sum_{i=1}^n \frac{M(R_i)}{X_i} + \int_V \frac{D(R)}{X} dv},$$

and

$$2.62 \quad Z_c = \frac{\sum_{i=1}^n \frac{M(R_i)}{X_i} \frac{Z_i}{X_i} + \int_V \frac{D(R)}{X} \frac{Z}{X} dv}{\sum_{i=1}^n \frac{M(R_i)}{X_i} + \int_V \frac{D(R)}{X} dv}.$$

The result obtained by combining equations 2.56 and 2.2 is

$$2.63 \quad M \frac{Y_c}{X_c} = \sum_{i=1}^n \frac{M(R_i)}{X_i} \frac{Y_i}{X_i} + \int_V \frac{D(R)}{X} \frac{Y}{X} dv.$$

This equation is satisfied if X_c , and Y_c are chosen by equations 2.60 and 2.61. Thus, it has been shown that the force field can be of type CII if and only if such mass distributions are chosen so as to satisfy the condition

$$2.64 \quad \sum_{i=1}^n \frac{M(R_i)}{X_i} + \int_V \frac{D(R)}{X} dv \neq 0.$$

Next, it is supposed that

$$2.65 \quad F(R) = (C_1, C_2, C_3).$$

This means that the force field is of type PI. Obviously equation 2.1 is satisfied by any \underline{R}_c , though equation 2.2 requires

$$2.66 \quad \underline{R}_c = \frac{\sum_{i=1}^M M(\underline{R}_i) + \int_V D(\underline{R}) \underline{R} \, dv}{M}.$$

This proves that $\underline{F}(\underline{R})$ can be of type PI.

Lastly, if $\underline{F}(\underline{R})$ is of type PII, equation 2.1 requires

$$2.67 \quad M \frac{1}{X_c} = \sum_{i=1}^M M(\underline{R}_i) \frac{1}{X_i} + \int_V D(\underline{R}) \frac{1}{X} \, dv,$$

or

$$2.68 \quad X_c = \frac{M}{\sum_{i=1}^M \frac{M(\underline{R}_i)}{X_i} + \int_V \frac{D(\underline{R})}{X} \, dv},$$

while 2.2 requires

$$2.69 \quad M \frac{Y_c}{X_c} = \sum_{i=1}^M M(\underline{R}_i) \frac{Y_i}{X_i} + \int_V D(\underline{R}) \frac{Y}{X} \, dv,$$

and

$$2.70 \quad M \frac{Z_c}{X_c} = \sum_{i=1}^M M(\underline{R}_i) \frac{Z_i}{X_i} + \int_V D(\underline{R}) \frac{Z}{X} \, dv.$$

Thus, since X_c must be chosen by equation 2.68, $\underline{F}(\underline{R})$ can be of type PII provided Y_c and Z_c are chosen to agree with 2.61 and 2.62, and provided

that only those mass distributions are accepted which satisfy conditions 2.64.

The following theorem has now been established.

Theorem 2.1. If $\underline{F}(\underline{R})$ satisfies the stated continuity conditions, a necessary and sufficient condition that there exist a point satisfying conditions i), ii) and iii) for an arbitrary mass distribution is that $\underline{F}(\underline{R})$ be one of the following four types.

CI:

$$2.71 \quad \underline{F}(\underline{R}) = A\underline{R},$$

$$2.72 \quad \underline{R}_c = \frac{\sum_{i=1}^n M(\underline{R}_i) \underline{R}_i + \int_V D(\underline{R}) \underline{R} dV}{M}.$$

This is a central field with center at (0,0,0).

CII:

$$2.73 \quad \underline{F}(\underline{R}) = B \left(\frac{X-D}{X}, \frac{Y}{X}, \frac{Z}{X} \right),$$

$$2.74 \quad \underline{R}_c = (X_c, Y_c, Z_c),$$

$$2.75 \quad X_c = \frac{M}{\sum_{i=1}^n \frac{M(\underline{R}_i)}{X_i} + \int_V \frac{D(\underline{R})}{X} dV},$$

$$2.76 \quad Y_c = \frac{\sum_{i=1}^n \frac{M(\underline{R}_i)}{X_i} \frac{Y_i}{X_i} + \int_V \frac{D(\underline{R})}{X} \frac{Y}{X} dv}{\sum_{i=1}^n \frac{M(\underline{R}_i)}{X_i} + \int_V \frac{D(\underline{R})}{X} dv},$$

$$Z_c = \frac{\sum_{i=1}^n \frac{M(\underline{R}_i)}{X_i} \frac{Z_i}{X_i} + \int_V \frac{D(\underline{R})}{X} \frac{Z}{X} dv}{\sum_{i=1}^n \frac{M(\underline{R}_i)}{X_i} + \int_V \frac{D(\underline{R})}{X} dv}.$$

This is a central field with center at (0,0,0).

PI:

$$2.78 \quad \underline{F}(\underline{R}) = (C_1, C_2, C_3),$$

$$2.79 \quad \underline{R}_c = \frac{\sum_{i=1}^n \frac{M(\underline{R}_i)}{X_i} \underline{R}_i + \int_V \frac{D(\underline{R})}{X} \underline{R} dv}{M}.$$

This is a parallel field.

PII:

$$2.80 \quad \underline{F}(\underline{R}) = \frac{1}{X} (D_1, D_2, D_3).$$

In this case the center of gravity is the same as for CII, and the field is a parallel field.

E. Summary of Results

The properties of the four types of fields are summarized as follows:

PI. If \underline{F} is of type PI, the forces are everywhere parallel and constant.

PII. If \underline{F} is of type PII the forces are again parallel, but they become infinite on an entire plane P . Elsewhere the forces are proportional to the distance from the plane P . Equal forces exist on planes parallel to P . It is to be remembered that in this case, only such mass distributions as will satisfy the relation 2.64 can be allowed. This condition will be satisfied if only such mass systems are considered as lie in that portion of E_3 that lies on one side of the plane P .

CI. If, however, \underline{F} is of type CI, then the forces all point toward a center θ , and are in magnitude proportional to the distance from θ . Forces of equal magnitude lie on circles with center at θ .

CII. In case \underline{F} is of type CII, the forces all point toward a common point θ called the center of the field and are infinite on an entire plane which is called P . At any point not in P the force is proportional to the distance from the point to the source and inversely proportional to the distance from the point to the plane P . Again, mass distributions must satisfy condition 2.64. This condition is clearly satisfied if only such mass distributions as lie entirely on one side of the plane P are considered. In this case the magnitude of \underline{F} is constant on surfaces of revolution defined by the equation

$$2.81 \quad (X-D)^2 + Y^2 + Z^2 = KX^2,$$

where K is a constant.

It is significant to note that, in cases PI and CII, the center of gravity of a mass system stays fixed in the system under arbitrary rigid body translations and rotations. However, in cases PII and CII the center of gravity stays fixed only for special translations and rotations. In the next chapter there will be investigated force fields for which a center of gravity can be defined which stays fixed in a mass system under arbitrary rigid body motion, but which need not satisfy condition ii).

III. FORCE FIELDS THAT ADMIT OF A CENTER OF GRAVITY IN A SECOND SENSE

A. A Second Definition of Center of Gravity

In this chapter a force field is said to admit of a center of gravity if, corresponding to any mass system placed in the force field, there exists a point called the center of gravity which satisfies the following two conditions.

1) The forces acting on a system have as a resultant the force which would act on a particle placed at the center of gravity, if this particle had a mass equal to the total mass of the system.

2) The center of gravity stays fixed in the system as the system undergoes arbitrary rigid body translations and rotations.

The problem is to find the most general class of functions which is continuous in some open region of E_3 , and for which there exists a center of gravity satisfying conditions 1) and 2) for arbitrary mass distributions.

B. Derivation of Fundamental Functional Equation

A mass distribution consisting of point-masses $M(\underline{R}_i)$, $i = 1, 2, \dots$, and of continuously distributed mass of density function $D(\underline{R})$ is now considered. Condition 1) requires that

$$3.1 \quad \sum_{i=1} M(\underline{R}_i) \underline{F}(\underline{R}_i) + \int_V D(\underline{R}) \underline{F}(\underline{R}) dv = \left(\sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) dv \right) \underline{F}(\underline{R}_c).$$

Of course \underline{R}_c must not only be such as to allow this functional equation to have a solution, but must also satisfy condition 2). The nature of \underline{F} , along a straight line, is first investigated.

C. Point-mass Distributions in E_3

1. Line distribution of particles

In this section it is assumed that the mass system consists of mass-particles only, and that these particles are distributed along the positive X-axis. The case in which there are just two particles of equal mass is first considered. Let these be $M(\underline{U}_1 X_1)$ and $M(\underline{U}_1 X_2)$. It is next shown that \underline{R}_c is given by

$$3.2 \quad \underline{R}_c = \underline{U}_1 \frac{X_1 + X_2}{2}.$$

To prove this, it is supposed that the center of gravity is at some other point. The mass system is then rotated so as to interchange the two mass-particles. Since the two masses are assumed equal the resulting mass system is in every way equivalent to the original with the exception of the fact that the center of gravity has been reflected through the point $\underline{U}_1 \frac{X_1 + X_2}{2}$. This completes the argument. Under these conditions,

equation 3.1 reduces to

$$3.3 \quad M(\underline{U}_1 X_1) \underline{F}(\underline{U}_1 X_1) + M(\underline{U}_1 X_2) \underline{F}(\underline{U}_1 X_2) = (M(\underline{U}_1 X_1) + M(\underline{U}_1 X_2)) \underline{F}(\underline{U}_1 \frac{X_1 + X_2}{2}).$$

But, since $M(\underline{U}_1 X_1) = M(\underline{U}_2 X_2)$, then

$$3.4 \quad \underline{F}(\underline{U}_1 X_1) + \underline{F}(\underline{U}_1 X_2) = 2 \underline{F}(\underline{U}_1 \frac{X_1 + X_2}{2}).$$

If

$$3.5 \quad \underline{F}(\underline{U}_1 X) = \underline{G}(X) = (G_1(X), G_2(X), G_3(X)),$$

then equation 3.4 becomes

$$3.6 \quad \underline{G}(X_1) + \underline{G}(X_2) = 2\underline{G}(\frac{X_1 + X_2}{2}).$$

Setting, first $X_1 = 0$, and then $X_2 = 0$, results in the two equations

$$3.7 \quad \underline{G}(X_1) = 2\underline{G}(\frac{X_1}{2}) - \underline{G}(0), \quad i = 1, 2.$$

Substituting from equation 3.7 into 3.6 gives

$$3.8 \quad 2\underline{G}(\frac{X_1}{2}) - 2\underline{G}(0) + 2\underline{G}(\frac{X_2}{2}) = 2\underline{G}(\frac{X_1 + X_2}{2}).$$

Subtracting $2\underline{G}(0)$ from both sides of equation 3.8 results in the equation

$$3.9 \quad 2\underline{G}(\frac{X_1}{2}) - 2\underline{G}(0) + 2\underline{G}(\frac{X_2}{2}) - 2\underline{G}(0) = 2\underline{G}(\frac{X_1 + X_2}{2}) - 2\underline{G}(0).$$

Dividing by two, setting $\frac{X_1}{2} = Z_1$, and replacing $\underline{G}(Z_1) - \underline{G}(0)$ by $\underline{H}(Z_1)$ gives

$$3.10 \quad \underline{H}(Z_1) + \underline{H}(Z_2) = \underline{H}(Z_1 + Z_2).$$

By taking Z_1 and Z_2 equal to zero, it is clear that

$$3.11 \quad \underline{H}(0) = 0.$$

If

$$3.12 \quad \underline{H}(Z) = (H_1(Z), H_2(Z), H_3(Z)),$$

then equation 3.10 is equivalent to

$$3.13 \quad H_i(Z_1) + H_i(Z_2) = H_i(Z_1 + Z_2), \quad i = 1, 2, 3.$$

This proves that $H_i(Z)$ must satisfy Cauchy's functional equation, and thus that

$$3.14 \quad H_i(Z) = A_i Z, \quad i = 1, 2, 3.$$

or

$$3.15 \quad \underline{H}(Z) = (A_1, A_2, A_3)Z = \underline{A}Z.$$

That is,

$$3.16 \quad \underline{G}\left(\frac{X}{2}\right) = \underline{A} \frac{X}{2} + \underline{G}(0),$$

or

$$3.17 \quad \underline{F}(\underline{U}_1 X) = \underline{A}X + \underline{B},$$

where \underline{A} and \underline{B} are constant vectors.

At this point it has been proved that if there exists a force field defined on the X -axis which is continuous on some open interval, and for which there exists a center of gravity which satisfies conditions 1) and 2) for arbitrary mass distributions, then, along the X -axis, the field is given by

$$3.18 \quad F(\underline{U}_1 X) = \underline{A}X + \underline{B}.$$

Clearly, \underline{F} , as defined in this way, satisfies the required continuity conditions. The force \underline{F} , as defined by 3.18, is now substituted into equation 3.1 for a mass system consisting of particles $M(\underline{U}_1 X_1)$, $i = 1, 2, 3, \dots$. This requires that

$$3.19 \quad \sum_{i=1} M(\underline{U}_1 X_1) (\underline{A}X_1 + \underline{B}) \equiv \sum_{i=1} M(\underline{U}_1 X_1) (\underline{A}X_c + \underline{B}).$$

Subtracting $\sum_{i=1} M(\underline{U}_1 X_1) \underline{B}$ from both sides of this identity gives

$$3.20 \quad \underline{A} \sum_{i=1} M(\underline{U}_1 X_1) X_1 = \underline{A} \sum_{i=1}^n M(\underline{U}_1 X_1) X_c.$$

This equation is satisfied identically if the center of gravity is given by

$$3.21 \quad X_c = \frac{\sum_{i=1} M(\underline{U}_1 X_1) X_1}{\sum_{i=1} M(\underline{U}_1 X_1)}.$$

2. Arbitrary distribution of mass-particles

In this section mass distributions consisting of arbitrary distributions of mass-particles are considered. However, a system consisting of two equal particles $M(\underline{R}_1)$ and $M(\underline{R}_2)$ is first studied. If \underline{R}_c , for this two particle system, is to satisfy condition 2) it must be

$$3.22 \quad \underline{R}_c = \frac{\underline{R}_1 + \underline{R}_2}{2},$$

for, as seen before, the mass system is left unaltered by a rotation that interchanges the two masses, while on the other hand only the point $(\underline{R}_1 + \underline{R}_2)/2$ remains fixed. Hence equation 3.1 becomes

$$3.23 \quad M(\underline{R}_1)\underline{F}(\underline{R}_1) + M(\underline{R}_2)\underline{F}(\underline{R}_2) = (M(\underline{R}_1) + M(\underline{R}_2))\underline{F}\left(\frac{\underline{R}_1 + \underline{R}_2}{2}\right).$$

But $M(\underline{R}_1) = M(\underline{R}_2)$ and thus equation 3.23 is equivalent to

$$3.24 \quad \underline{F}(\underline{R}_1) + \underline{F}(\underline{R}_2) = 2\underline{F}\left(\frac{\underline{R}_1 + \underline{R}_2}{2}\right).$$

Setting first $\underline{R}_1 = 0$ and then $\underline{R}_2 = 0$ results in

$$3.25 \quad \underline{F}(\underline{R}_1) = 2\underline{F}\left(\frac{\underline{R}_1}{2}\right) - \underline{F}(0),$$

and

$$3.26 \quad \underline{F}(\underline{R}_2) = 2\underline{F}\left(\frac{\underline{R}_2}{2}\right) - \underline{F}(0).$$

The last three equations can be combined to give

$$3.27 \quad 2\underline{F}\left(\frac{\underline{R}_1}{2}\right) + 2\underline{F}\left(\frac{\underline{R}_2}{2}\right) - 2\underline{F}(0) = 2\underline{F}\left(\frac{\underline{R}_1 + \underline{R}_2}{2}\right),$$

or

$$3.28 \quad \underline{F}\left(\frac{\underline{R}_1}{2}\right) - \underline{F}(0) + \underline{F}\left(\frac{\underline{R}_2}{2}\right) - \underline{F}(0) = \underline{F}\left(\frac{\underline{R}_1 + \underline{R}_2}{2}\right) - \underline{F}(0).$$

Next $\frac{\underline{R}_1}{2}$ is replaced by \underline{T}_1 and $\underline{F}(\underline{T}_1) - \underline{F}(0)$ by $\underline{H}(\underline{T}_1)$. Equation 3.28 then is

$$3.29 \quad \underline{H}(\underline{T}_1) + \underline{H}(\underline{T}_2) = \underline{H}(\underline{T}_1 + \underline{T}_2).$$

This can be considered as a further generalization of Cauchy's functional equation. If \underline{T}_1 and \underline{T}_2 are set equal to zero then equation 3.29 becomes

$$3.30 \quad 2\underline{H}(0) = \underline{H}(0),$$

or

$$3.31 \quad \underline{H}(0) = 0.$$

If in equation 3.29, \underline{T}_2 is replaced by $\underline{T}_2 + \underline{T}_3$, then this equation becomes

$$3.32 \quad \underline{H}(\underline{T}_1) + \underline{H}(\underline{T}_2 + \underline{T}_3) = \underline{H}(\underline{T}_1 + \underline{T}_2 + \underline{T}_3).$$

Using the property 3.29, this is now written

$$3.33 \quad H(\underline{T}_1) + H(\underline{T}_2) + H(\underline{T}_3) = H(\underline{T}_1 + \underline{T}_2 + \underline{T}_3).$$

If one sets

$$3.34 \quad H(\underline{T}_k) = (H_1(T_{1k}, T_{2k}, T_{3k}), H_2(T_{1k}, T_{2k}, T_{3k}), H_3(T_{1k}, T_{2k}, T_{3k})),$$

equation 3.33 becomes

$$3.35 \quad \begin{aligned} & H_1(T_{11}, T_{21}, T_{31}) + H_1(T_{12}, T_{22}, T_{32}) + H_1(T_{13}, T_{23}, T_{33}) \\ & = H_1(T_{11}+T_{21}+T_{31}, T_{12}+T_{22}+T_{32}, T_{13}+T_{23}+T_{33}), \quad i = 1, 2, 3. \end{aligned}$$

Next, T_{jk} is taken equal to zero for each j and k such that $j + k \neq 4$, in order to show that

$$3.36 \quad H_1(0, 0, T_{31}) + H_1(0, T_{22}, 0) + H_1(T_{13}, 0, 0) = H_1(T_{13}, T_{22}, T_{31}).$$

Returning to equation 3.35 and setting

$$T_{21} = T_{31} = T_{22} = T_{32} = T_{13} = T_{23} = T_{33} = 0,$$

gives

$$3.37 \quad H_1(T_{11}, 0, 0) + H_1(T_{12}, 0, 0) = H_1(T_{11}+T_{12}, 0, 0), \quad i = 1, 2, 3.$$

Setting $T_{11} = T_{31} = T_{12} = T_{32} = T_{13} = T_{23} = T_{33} = 0$ in equation

3.35 gives

$$3.38 \quad H_1(0, T_{21}, 0) + H_1(0, T_{22}, 0) = H_1(0, T_{21} + T_{22}, 0), \quad i = 1, 2, 3.$$

Or finally, if $T_{11} = T_{21} = T_{12} = T_{22} = T_{13} = T_{23} = T_{33} = 0$, then

$$3.39 \quad H_1(0, 0, T_{31}) + H_1(0, 0, T_{32}) = H_1(0, 0, T_{31} + T_{32}), \quad i = 1, 2, 3.$$

Each of the above nine functions must satisfy Cauchy's functional equation.

Hence their solutions are

$$3.40 \quad H_1(T_{1k}, 0, 0) = A_{11}T_{1k}, \quad i = 1, 2, 3,$$

$$3.41 \quad H_1(0, T_{2k}, 0) = A_{12}T_{2k}, \quad i = 1, 2, 3,$$

and

$$3.42 \quad H_1(0, 0, T_{3k}) = A_{13}T_{3k}, \quad i = 1, 2, 3.$$

Adding equations 3.40, 3.41 and 3.42, in turn, for $i = 1, 2, 3$, results in

$$3.43 \quad H_1(T_{1k}, 0, 0) + H_1(0, T_{2k}, 0) + H_1(0, 0, T_{3k}) = A_{11}T_{1k} + A_{12}T_{2k} + A_{13}T_{3k}, \quad i = 1, 2, 3,$$

which with the help of equation 3.36 establishes the fact that

$$3.44 \quad H_1(T_{1k}, T_{2k}, T_{3k}) = \sum_{j=1}^3 A_{1j}T_{jk}.$$

Or, by using the definitions of H_i and T_{sk} , one obtains

$$\underline{H}(\underline{T}) = \underline{A} \cdot \underline{T},$$

where

$$3.45 \quad A = \sum_{i=1}^3 \sum_{j=1}^3 \underline{U}_i \underline{U}_j A_{ij}.$$

It is to be remembered that

$$3.46 \quad \underline{U}_i \cdot \underline{U}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

By using the definition of \underline{H} and \underline{T} , one obtains

$$3.47 \quad \underline{F}(\underline{R}) = \bar{\underline{A}} \cdot \underline{R} + \underline{F}(0) = \bar{\underline{A}} \cdot \underline{R} + \underline{B},$$

where $\bar{\underline{A}}$ is a dyadic and \underline{B} is a vector.

It has been shown that, if conditions 1), and 2) along with the stated continuity conditions are to be satisfied for an arbitrary distribution of mass-particles, then \underline{F} must be given by equation 3.47. Clearly, \underline{F} , as defined in this way, satisfies the desired continuity conditions, but it remains to be shown that conditions 1), and 2) are actually satisfied. Substituting this \underline{F} into equation 3.1 for a mass system consisting of mass-particles $M(\underline{R}_i)$, $i = 1, 2, \dots$, gives

$$3.48 \quad \sum_{i=1} M(\underline{R}_i) (\bar{\underline{A}} \cdot \underline{R}_i + \underline{B}) = \sum_{i=1} M(\underline{R}_i) (\bar{\underline{A}} \cdot \underline{R}_c + \underline{B}).$$

Subtracting $\sum_{i=1} M(\underline{R}_i) \underline{B}$ from both sides of this identity, results in the

equivalent identity.

$$3.49 \quad \bar{A} \cdot \left(\sum_{i=1} M(\underline{R}_i) \underline{R}_i - \sum_{i=1} M(\underline{R}_i) \underline{R}_c \right) = 0.$$

Thus condition 1) will be satisfied if

$$3.50 \quad \underline{R}_c = \frac{\sum_{i=1} M(\underline{R}_i) \underline{R}_i}{\sum_{i=1} M(\underline{R}_i)}.$$

It is to be observed that \underline{R}_c , as defined by equation 3.50, also satisfies condition 2).

The following theorem has been established.

Theorem 3.1. If $\underline{F}(\underline{R})$ is continuous in some open region of E_3 , then a necessary and sufficient condition that there exist a point satisfying conditions 1) and 2) for arbitrary point-mass distributions is that $\underline{F}(\underline{R})$ be given by the equation

$$3.51 \quad \underline{F}(\underline{R}) = \bar{A} \cdot \underline{R} + \underline{B},$$

where \bar{A} is an arbitrary constant dyadic, and \underline{B} is an arbitrary constant vector. In this case the center of gravity may be taken as

$$3.52 \quad \underline{R}_c = \frac{\sum_{i=1} M(\underline{R}_i) \underline{R}_i}{\sum_{i=1} M(\underline{R}_i)}.$$

D. Arbitrary Mass Distributions in E_3

It has been shown that if conditions 1) and 2) along with the stated

continuity conditions are to be satisfied for arbitrary mass distribution, then \underline{F} must be given by equation 3.51. It is next shown that if \underline{F} is given by 3.51 then these conditions are satisfied. The stated continuity conditions are clearly satisfied. Condition 1) is checked by substituting $\underline{F}(\underline{R})$ from equation 3.51 into equation 3.1, and then establishing the identity that results. This identity is

$$3.53 \quad \sum_{i=1} M(\underline{R}_i) (\bar{\underline{A}} \cdot \underline{R}_i + \underline{B}) + \int_V D(\underline{R}) (\bar{\underline{A}} \cdot \underline{R} + \underline{B}) d\underline{v} = \left(\sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) d\underline{v} \right) (\bar{\underline{A}} \cdot \underline{R}_c + \underline{B}).$$

This identity is equivalent to

$$3.54 \quad \bar{\underline{A}} \cdot \left(\sum_{i=1} M(\underline{R}_i) \underline{R}_i + \int_V D(\underline{R}) \underline{R} d\underline{v} \right) = \left(\sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) d\underline{v} \right) \bar{\underline{A}} \cdot \underline{R}_c,$$

or

$$3.55 \quad \bar{\underline{A}} \cdot \left(\underline{R}_c - \frac{\sum_{i=1} M(\underline{R}_i) \underline{R}_i + \int_V D(\underline{R}) \underline{R} d\underline{v}}{\sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) d\underline{v}} \right) = 0.$$

Thus if

$$3.56 \quad \underline{R}_c = \frac{\sum_{i=1} M(\underline{R}_i) \underline{R}_i + \int_V D(\underline{R}) \underline{R} d\underline{v}}{\sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) d\underline{v}},$$

then conditions 1) and 2) are satisfied for arbitrary $\bar{\underline{A}}$, and \underline{B} .

This completes the proof of the following theorem.

Theorem 3.2 If $\underline{F}(\underline{R})$ is continuous in some open region of E_3 , then a necessary and sufficient condition that there exist a point satisfying conditions 1) and 2) for arbitrary mass distribution is that $\underline{F}(\underline{R})$ be given by the equation

$$3.57 \quad \underline{F}(\underline{R}) = \bar{\underline{A}} \cdot \underline{R} + \underline{B}$$

where $\bar{\underline{A}}$ is an arbitrary constant dyadic, and \underline{B} an arbitrary constant vector. In this case the center of gravity may be taken as

$$3.58 \quad \underline{R} = \frac{\sum_{i=1}^n M(\underline{R}_i)(\underline{R}_i) + \int_V D(\underline{R})\underline{R}dV}{\sum_{i=1}^n M(\underline{R}_i) + \int_V D(\underline{R})dV}.$$

If $\bar{\underline{A}}$ is a scalar K times the idemfactor, then equation 3.57 takes a form

$$3.59 \quad \underline{F}(\underline{R}) = K\underline{R} + \underline{B},$$

and, if in this equation \underline{R} is replaced by $\underline{R} - \underline{B}/K$, then

$$3.60 \quad \underline{F}'(\underline{R}) = K\underline{R},$$

where
$$\underline{F}'(\underline{R}) = \underline{F}(\underline{R} - \frac{\underline{B}}{K}).$$

This shows that the force field given by 3.59 is central with source at $-\underline{B}/K$.

If $\bar{\underline{A}} = 0$, the force field is a constant parallel field. However, this is not the only parallel field that can be obtained from this rather general class of fields. In fact, if $\underline{B} \neq 0$, then $\underline{F}(\underline{R})$ is a parallel

field, if and only if,

$$3.61 \quad \underline{B} \times (\underline{\bar{A}} \cdot \underline{R}) \equiv 0.$$

As an example, this equation is satisfied for

$$3.62 \quad \underline{B} = (B_1, B_2, B_3),$$

and

$$3.63 \quad \underline{\bar{A}} = \sum_{j=1}^3 \sum_{i=1}^3 \underline{U}_i \underline{U}_j \underline{B}_i.$$

On the other hand, suppose $A_{11} = 1$, but otherwise $A_{ij} = 0$, and suppose $\underline{B} = \underline{U}_2$. Then

$$3.64 \quad \underline{F}(\underline{R}) = \underline{U}_1 X + \underline{U}_2.$$

On the line $Y = Z = 0$, the forces are central, but in the plane $X = 0$, the forces are parallel.

Thus, it is seen that force fields which admit of a center of gravity in the sense of this chapter include parallel fields, central fields, and fields that are neither parallel nor central.

IV. SUMMARY

In Chapter II it was proved (by a method outlined by Aczél and Fenyő) that, if \underline{F} is a force field which satisfies the stated continuity conditions, and for which there exists a center of gravity satisfying conditions i), ii), and iii), then $\underline{F}(\underline{R})$ is one of the four types:

$$4.1 \quad \underline{F}(\underline{R}) = A\underline{R}, \quad \text{CI,}$$

$$4.2 \quad \underline{F}(\underline{R}) = B\left(\frac{\underline{X}-\underline{D}}{\underline{X}}, \frac{\underline{Y}}{\underline{X}}, \frac{\underline{Z}}{\underline{X}}\right), \quad \text{CII,}$$

$$4.3 \quad \underline{F}(\underline{R}) = (C_1 \ C_2 \ C_3), \quad \text{PI,}$$

or

$$4.4 \quad \underline{F}(\underline{R}) = \frac{1}{\underline{X}} (D_1, D_2, D_3), \quad \text{PII.}$$

Conversely it was proved that, if \underline{F} is one of these four types, then it satisfies the stated continuity conditions and there exists a center of gravity which satisfies conditions i), ii), and iii). Of these four types, CI and PI are the only ones that satisfy conditions 1) and 2) along with the desired continuity conditions of Chapter III.

In Chapter III it was proved that, if \underline{F} is a force field (continuous in some open region of E_3) for which there exists a center of gravity satisfying conditions 1) and 2), then $\underline{F}(\underline{R})$ is of the form

$$4.5 \quad \underline{F}(\underline{R}) = \bar{A} \cdot \underline{R} + \underline{B} \quad G_1$$

and conversely. For convenience, this case is designated by G_1 . If \bar{A} is a scalar A times the idemfactor and if the origin is translated by replacing \underline{R} by $\underline{R} - \underline{B}/A$, then G_1 takes the form

$$4.6 \quad \underline{F}(\underline{R}) = A\underline{R}. \quad CI.$$

Next, if $\bar{A} = 0$, and $\underline{B} = (C_1, C_2, C_3)$, G_1 is

$$4.7 \quad \underline{F}(\underline{R}) = (C_1, C_2, C_3).$$

This shows that CI and PI are special cases of G_1 . However, it is clear that there are no other special cases of G_1 which satisfy conditions i), ii), and iii), for CI and PI are the only functions that remain bounded and still satisfy these last conditions.

A center of gravity is now defined in a third way. A force field is said to admit of a center of gravity in this third sense if, for an arbitrary mass distribution, there exists a point called the center of gravity for which the following four conditions are satisfied.

a) The forces acting on a mass system have as a resultant the force which would act on a particle placed at the center of gravity, if this particle had a mass equal to the total mass of the system.

b) The center of gravity stays fixed in a mass system as the system undergoes arbitrary rigid body translations and rotations.

c) A mass system, if thought of as rigid, has no tendency to rotate about its center of gravity.

d) The center of gravity of two point mass-particles lies on the line joining the particles.

The following theorem has been proved.

Theorem 4.1 If \underline{F} satisfies the continuity conditions stated in Chapter III, then a necessary and sufficient condition that there exist a point satisfying conditions a), b), c) and d) is that $\underline{F}(\underline{R})$ be

$$4.8 \quad \underline{F}(\underline{R}) = A\underline{R} + \underline{B},$$

where A is an arbitrary scalar, and \underline{B} is an arbitrary vector. In this case the center of gravity is

$$4.9 \quad \underline{R}_c = \frac{\sum_{i=1}^n M(\underline{R}_i)\underline{R}_i + \int_V D(\underline{R})\underline{R}dV}{\sum_{i=1}^n M(\underline{R}_i) + \int_V D(\underline{R})dV}$$

In order to prove that the condition d) and the stringent continuity conditions of this theorem can be eliminated, a mass system consisting of two identical particles located at the points \underline{R}_1 , and $2\underline{R}_1$ is now considered. As previously demonstrated, if this mass system is to satisfy condition b) above, then \underline{R}_c must be given by

$$4.10 \quad \underline{R}_c = \frac{\underline{R}_1 + 2\underline{R}_1}{2} = \frac{3}{2} \underline{R}_1.$$

Next, \underline{F} is taken as

$$4.11 \quad \underline{F}(\underline{R}) = \bar{A}\underline{R} + \underline{B}.$$

It is desired to find conditions on \bar{A} and \underline{B} that will allow condition c) to be satisfied. This condition requires that

$$4.12 \quad (\bar{A} \cdot \underline{R}_1 + \underline{B}) \times \underline{R}_1 + (\bar{A} \cdot 2\underline{R}_1 + \underline{B}) \times 2\underline{R}_1 = 2\bar{A} \cdot \left(\frac{3}{2} \underline{R}_1 + \underline{B}\right) \times \frac{3}{2} \underline{R}_1$$

Subtracting $3\underline{B} \times \underline{R}_1$ from both sides shows that the identity 4.12 is equivalent to

$$4.13 \quad (\bar{A} \cdot \underline{R}_1) \times \underline{R}_1 + 4(\bar{A} \cdot \underline{R}_1) \times \underline{R}_1 = \frac{9}{2} (\bar{A} \cdot \underline{R}_1) \times \underline{R}_1.$$

In turn, this is equivalent to

$$4.14 \quad (\bar{A} \cdot \underline{R}_1) \times \underline{R}_1 = 0.$$

Hence it has been shown that, if condition C) is to be satisfied for this mass system, then

$$4.15 \quad \bar{A} \cdot \underline{R}_1 = A \underline{R}_1,$$

where A is a scalar. But if

$$4.16 \quad \underline{F}(\underline{R}) = A \underline{R}_1 + \underline{B},$$

and, if

$$4.17 \quad \underline{R}_c = \frac{\sum_{i=1} M(\underline{R}_i) \underline{R}_i + \int_V D(\underline{R}) \underline{R} dv}{\sum_{i=1} M(\underline{R}_i) + \int_V D(\underline{R}) dv},$$

then, as shown earlier, condition c) is satisfied for arbitrary mass distributions.

This result along with Theorem 3.2 establishes the following theorem.

Theorem 4.2. If $\underline{F}(\underline{R})$ is continuous in some open interval of E_3 , then a necessary and sufficient condition that there exist a point which satisfies conditions a), b), and c) is that $\underline{F}(\underline{R})$ be given by the equation

$$4.18 \quad \underline{F}(\underline{R}) = A\underline{R} + \underline{B},$$

where A is an arbitrary scalar and \underline{B} is an arbitrary vector. In this case the center of gravity may be chosen as

$$4.19 \quad \underline{R}_c = \frac{\sum_{i=1}^n M(\underline{R}_i) \underline{R}_i + \int_V D(\underline{R}) \underline{R} dv}{\sum_{i=1}^n M(\underline{R}_i) + \int_V D(\underline{R}) dv}.$$

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